BOUNDARY VALUE PROBLEMS WITH MEASURES FOR ELLIPTIC EQUATIONS WITH SINGULAR POTENTIALS

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Abstract

We study the boundary value problem with Radon measures for nonnegative solutions of $-\Delta u + Vu = 0$ in a bounded smooth domain Ω, when V is a locally bounded nonnegative function. Introducing some specific capacity, we give sufficient conditions on a Radon measure μ on $\partial\Omega$ so that the problem can be solved. We study the reduced measure associated to this equation as well as the boundary trace of positive solutions.

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Contents

1 Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and V a locally bounded real valued measurable function defined in Ω. The first question we adress is the solvability of the following non-homogeneous Dirichlet problem with a Radon measure for boundary data,

$$\left\{ \begin{aligned} -\Delta u + Vu &= 0 && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega. \end{aligned} \right. \tag{1.1}$$

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Let ρ be the first (and positive) eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$. By a solution we mean a function $u \in L^1(\Omega)$, such that $Vu \in L^1_\rho$, which satisfies

$$\int_\Omega (-u\Delta\zeta + Vu\zeta)\, dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu. \tag{1.2}$$

for any function $\zeta \in C_0^1(\overline{\Omega})$ such that $\Delta\zeta \in L^\infty(\Omega)$. When V is a bounded nonnegative function, it is straightforward that there exist a unique solution. However, it is less obvious to find general conditions which allow the solvability for any $\mu \in \mathfrak{M}(\partial\Omega)$, the set of Radon measures on $\partial\Omega$. In order to avoid difficulties due to Fredholm type obstructions, we shall most often assume that V is nonnegative, in which case there exists at most one solution.

Let us denote by K^Ω the Poisson kernel in Ω and by $\mathbb{K}[\mu]$ the Poisson potential of a measure, that is

$$\mathbb{K}[\mu](x) := \int_{\partial\Omega} K^\Omega(x,y) d\mu(y) \qquad \forall x \in \Omega. \tag{1.3}$$

We first observe that, when $V \geq 0$ and the measure μ satisfies

$$\int_\Omega \mathbb{K}[|\mu|](x) V(x)\rho(x) dx < \infty, \tag{1.4}$$

then problem (1.1) admits a solution. A Radon measure which satisfies (1.4) is called *an admissible measure* and a measure for which a solution exists is called *a good measure*.

We first consider the *subcritical case* which means that the boundary value is solvable for any $\mu \in \mathfrak{M}(\partial\Omega)$. As a first result, we prove that any measure μ is admissible if V is nonnegative and satisfies

$$\sup_{y\in\partial\Omega} \operatorname{ess} \int_\Omega K^\Omega(x,y)V(x)\rho(x)dx < \infty. \tag{1.5}$$

Using estimates on the Poisson kernel, this condition is fulfilled if there exists $M > 0$ such that for any $y \in \partial\Omega$,

$$\int_0^{D(\Omega)} \left(\int_{\Omega\cap B_r(y)} V(x)\rho^2(x)dx \right) \frac{dr}{r^{N+1}} \leq M \tag{1.6}$$

where $D(\Omega) = diam(\Omega)$. We give also sufficient conditions which ensures that the boundary value problem (1.1) is stable from the weak*-topology of $\mathfrak{M}(\partial\Omega)$ to $L^1(\Omega) \cap L^1_{V\rho}(\Omega)$. One of the sufficient conditions is that $V \geq 0$ satisfies

$$\lim_{\epsilon\to 0} \int_0^\epsilon \left(\int_{\Omega\cap B_r(y)} V(x)\rho^2(x)dx \right) \frac{dr}{r^{N+1}} = 0, \tag{1.7}$$

uniformly with respect to $y \in \partial\Omega$.

In the *supercritical case* problem (1.1) cannot be solved for any $\mu \in \mathfrak{M}(\partial\Omega)$. In order to characterize positive good measures, we introduce a framework of nonlinear analysis which have been used by Dynkin and Kuznetsov (see [9] and references therein) and Marcus and Véron [16] in their study of the boundary value problems with measures

$$\begin{cases} -\Delta u + |u|^{q-1}u = 0 & \text{in } \Omega \\ u = \mu & \text{in } \partial\Omega, \end{cases} \tag{1.8}$$

where $q > 1$. In these works, positive good measures on $\partial\Omega$ are completely characterized by the $C_{2/q,q'}$-Bessel in dimension N-1 and the following property:

A measure $\mu \in \mathfrak{M}_+(\partial\Omega)$ is good for problem (1.8) *if and only if it does charge Borel sets with zero $C_{2/q,q'}$-capacity, i.e*

$$C_{2/q,q'}(E) = 0 \Longrightarrow \mu(E) = 0 \qquad \forall E \subset \partial\Omega,\ E \text{ Borel.} \tag{1.9}$$

Moreover, any positive good measure is the limit of an increasing sequence $\{\mu_n\}$ of admissible measures which, in this case, are the positive measures belonging to the Besov space $B_{2/q,q'}(\partial\Omega)$. They also characaterize removable sets in terms of $C_{2/q,q'}$-capacity.

In our present work, and always with $V \geq 0$, we use a capacity associated to the Poisson kernel K^Ω and belongs to a class studied by Fuglede [10] [11]. It is defined by

$$C_V(E) = \sup\{\mu(E) : \mu \in \mathfrak{M}_+(\partial\Omega), \mu(E^c) = 0,\ \|V\mathbb{K}[\mu]\|_{L^1_\rho} \leq 1\}, \tag{1.10}$$

for any Borel set $E \subset \partial\Omega$. Furtheremore $C_V(E)$ is equal to the value of its dual expression $C^*_V(E)$ defined by

$$C^*_V(E) = \inf\{\|f\|_{L^\infty} : \check{\mathbb{K}}[f] \geq 1 \quad \text{on } E\}, \tag{1.11}$$

where

$$\check{\mathbb{K}}[f](y) = \int_\Omega K^\Omega(x,y) f(x) V(x) \rho(x) dx \qquad \forall y \in \partial\Omega. \tag{1.12}$$

If E is a compact subset of $\partial\Omega$, this capacity is explicitely given by

$$C_V(E) = C^*_V(E) = \max_{y\in E}\left(\int_\Omega K^\Omega(x,y)V(x)\rho(x)dx\right)^{-1}. \tag{1.13}$$

We denote by Z_V the largest set with zero C_V capacity, i.e.

$$Z_V = \left\{y \in \partial\Omega : \int_\Omega K^\Omega(x,y)V(x)\rho(x)dx = \infty\right\}, \tag{1.14}$$

and we prove the following.

1- If $\{\mu_n\}$ is an increasing sequence of positive good measures which converges to a measure μ in the weak topology, then μ is a good measure.*

2- If $\mu \in \mathfrak{M}_+(\partial\Omega)$ satisfies $\mu(Z_V) = 0$, then μ is a good measure.

3- A good measure μ vanishes on Z_V if and only if there exists an increasing sequence of positive admissible measures which converges to μ in the weak topology.*

In section 4 we study relaxation phenomenon in replacing (1.1) by the truncated problem

$$\left\{\begin{array}{rl} -\Delta u + V_k u = 0 & \quad \text{in } \Omega \\ u = \mu & \quad \text{in } \partial\Omega. \end{array}\right. \tag{1.15}$$

where $\{V_k\}$ is an increasing sequence of positive bounded functions which converges to V locally uniformly in Ω. We adapt to the linear problem some of the principles of the reduced measure.

This notion is introduced by Brezis, Marcus and Ponce [5] in the study of the nonlinear Poisson equation

$$-\Delta u+g(u)=\mu \qquad \text{in } \Omega \tag{1.16}$$

and extended to the Dirichlet problem

$$\left\{\begin{array}{ll} -\Delta u+g(u)=0 & \text{in } \Omega \\ u=\mu & \text{in } \partial\Omega, \end{array}\right. \tag{1.17}$$

by Brezis and Ponce [6]. In our construction, problem (1.15) admits a unique solution u_k. The sequence $\{u_k\}$ decreases and converges to some u which satisfies a relaxed boundary value problem

$$\left\{\begin{array}{ll} -\Delta u+Vu=0 & \text{in } \Omega \\ u=\mu^* & \text{in } \partial\Omega. \end{array}\right. \tag{1.18}$$

The measure μ^* is called the *reduced measure* associated to μ and V. *Note that* μ^* *is the largest measure for which the problem*

$$\left\{\begin{array}{ll} -\Delta u+Vu=0 & in\ \Omega \\ u=\nu\leq\mu & in\ \partial\Omega. \end{array}\right. \tag{1.19}$$

admits a solution. This truncation process allows to construct the Poisson kernel K_V^{Ω} associated to the operator $-\Delta+V$ as being the limit of the decreasing limit of the sequence of kernel functions $\{K_{V_k}^{\Omega}\}$ asociated to $-\Delta+V_k$. The solution $u=u_{\mu^*}$ of (1.18) is expressed by

$$u_{\mu^*}(x)=\int_{\partial\Omega}K_V^{\Omega}(x,y)d\mu(y)=\int_{\partial\Omega}K_V^{\Omega}(x,y)d\mu^*(y) \qquad \forall x\in\Omega. \tag{1.20}$$

We define the vanishing set of K_V by

$$Z_V^*=\{y\in\partial\Omega : K_V^{\Omega}(x_0,y)=0\}, \tag{1.21}$$

for some $x_0\in\Omega$, and thus for any $x\in\Omega$ by Harnack inequality. We prove

1- $Z_V^*\subset Z_V$.

2- $\mu^*=\mu\chi_{Z_V^*}$.

A challenging open problem is to give conditions on V which allows $Z_V^*=Z_V$.

The last section is devoted to the construction of the boundary trace of positive solutions of

$$-\Delta u+Vu=0 \qquad \text{in } \Omega, \tag{1.22}$$

assuming $V\geq 0$. Using results of [18], we defined the regular set $\mathcal{R}(u)$ of the boundary trace of u. This set is a relatively open subset of $\partial\Omega$ and the regular part of the boundary trace is represented by a positive Radon measure μ_u on $\mathcal{R}(u)$. In order to study the singular set of the boundary trace $\mathcal{S}(u):=\partial\Omega\setminus\mathcal{R}(u)$, we adapt the sweeping method introduced by Marcus and Véron in [19] for equation

$$-\Delta u+g(u)=0 \qquad \text{in } \Omega. \tag{1.23}$$

If μ is a good positive measure concentrated on $\mathcal{S}(u)$, and u_μ is the unique solution of (1.1) with boundary data μ, we set $v_\mu = \min\{u, u_\mu\}$. Then v_μ is a positive super solution which admits a positive trace $\gamma_u(\mu) \in \mathfrak{M}_+(\partial\Omega)$. The extended boundary trace $Tr^e(u)$ of u is defined by

$$\nu(u)(E) := Tr^e(u)(E) = \sup\{\gamma_u(\mu)(E) : \mu \text{ good}, E \subset \partial\Omega,\ E \text{ Borel}\}. \tag{1.24}$$

Then $Tr^e(u)$ is a Borel measure on Ω. If we assume moreover that

$$\lim_{\epsilon\to 0}\int_0^\epsilon \left(\int_{\Omega\cap B_r(y)} V(x)\rho^2(x)dx\right)\frac{dr}{r^{N+1}} = 0 \qquad \textit{uniformly with respect to } y \in \partial\Omega, \tag{1.25}$$

then $Tr^e(u)$ is a bounded measure and therefore a Radon measure. Finally, if $N = 2$ and (1.25) holds, or if $N = 2$ and there holds

$$\lim_{\epsilon\to 0}\int_0^\epsilon \left(\int_{\Omega\cap B_r(y)} V(x)(\rho(x)-\epsilon)_+^2 dx\right)\frac{dr}{r^{N+1}} = 0, \tag{1.26}$$

uniformly with respect to $\epsilon \in (0, \epsilon_0]$ and y s.t. $\operatorname{dist}(x, \partial\Omega) = \epsilon$, then $u = u_{\nu(u)}$.

If $V(x) \le v(\rho(x)$ for some v which satisfies

$$\int_0^1 v(t)tdt < \infty, \tag{1.27}$$

then Marcus and Véron proved in [18] that $u = u_{\nu_u}$. Actually, when V has such a geometric form, the assumptions (1.25)-(1.26) and (1.27) are equivalent.

2 The subcritical case

In the sequel Ω is a bounded smooth domain in $\mathbb{R}^N$ and $V \in L^\infty_{loc}$. We denote by ρ the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$, $\rho > 0$ with the corresponding eigenvalue λ, by $\mathfrak{M}(\partial\Omega)$ the space of bounded Radon measures on $\partial\Omega$ and by $\mathfrak{M}_+(\partial\Omega)$ its positive cone. For any positive Radon measure on $\partial\Omega$, we shall denote by the same symbol the corresponding outer regular bounded Borel measure. Conversely, for any outer regular bounded Borel μ, we denote by the same expression μ the Radon measure defined on $C(\partial\Omega)$ by

$$\zeta \mapsto \mu(\zeta) = \int_{\partial\Omega} \zeta d\mu.$$

If $\mu \in \mathfrak{M}(\partial\Omega)$, we are concerned with the following problem

$$\left\{\begin{aligned} -\Delta u + Vu &= 0 && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega. \end{aligned}\right. \tag{2.1}$$

Definition 2.1 *Let $\mu \in \mathfrak{M}(\partial\Omega)$. We say that u is a weak solution of (2.1), if $u \in L^1(\Omega)$, $Vu \in L^1_\rho(\Omega)$ and, for any $\zeta \in C_0^1(\overline{\Omega})$ with $\Delta\zeta \in L^\infty(\Omega)$, there holds*

$$\int_\Omega (-u\Delta\zeta + Vu\zeta)\,dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu. \tag{2.2}$$

In the sequel we put

$$T(\Omega) := \{\zeta \in C_0^1(\overline{\Omega}) \text{ such that } \Delta\zeta \in L^\infty(\Omega)\}.$$

We recall the following estimates obtained by Brezis [4]

Proposition 2.2 *Let $\mu \in L^1(\partial\Omega)$ and u be a weak solution of problem* (2.1). *Then there holds*

$$\|u\|_{L^1(\Omega)} + \|V_+u\|_{L^1_\rho(\Omega)} \leq \|V_-u\|_{L^1_\rho(\Omega)} + c\,\|\mu\|_{L^1(\partial\Omega)} \tag{2.3}$$

$$\int_\Omega (-|u|\Delta\zeta + V|u|\zeta)\,dx \leq -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}}|\mu|dS \tag{2.4}$$

and

$$\int_\Omega (-u_+\Delta\zeta + Vu_+\zeta)\,dx \leq -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}}\mu_+dS, \tag{2.5}$$

for all $\zeta \in T(\Omega)$, $\zeta \geq 0$.

We denote by $K^\Omega(x,y)$ the Poisson kernel in Ω and by $\mathbb{K}[\mu]$ the Poisson potential of $\mu \in \mathfrak{M}(\partial\Omega)$ defined by

$$\mathbb{K}[\mu](x) = \int_{\partial\Omega} K^\Omega(x,y)d\mu(y) \qquad \forall x \in \Omega. \tag{2.6}$$

Definition 2.3 *A measure μ on $\partial\Omega$ is* **admissible** *if*

$$\int_\Omega \mathbb{K}[|\mu|](x)|V(x)|\rho(x)dx < \infty. \tag{2.7}$$

It is **good** *if problem* (2.1) *admits a weak solution.*

We notice that, if there exists at least one admissible positive measure μ, then

$$\int_\Omega V(x)\rho^2(x)dx < \infty. \tag{2.8}$$

Theorem 2.4 *Assume $V \geq 0$, then problem* (2.1) *admits at most one solution. Furthermore, if μ is admissible, then there exists a unique solution that we denote u_μ.*

Proof. Uniqueness follows from (2.3). For existence we can assume $\mu \geq 0$. For any $k \in \mathbb{N}_*$ set $V_k = \inf\{V,k\}$ and denote by $u := u_k$ the solution of

$$\left\{\begin{array}{ll} -\Delta u + V_k(x)u = 0 & \text{in } \Omega \\ u = \mu & \text{on } \partial\Omega. \end{array}\right. \tag{2.9}$$

Then $0 \leq u_k \leq \mathbb{K}[\mu]$. By the maximum principle, u_k is decreasing and converges to some u, and

$$0 \leq V_ku_k \leq V\mathbb{K}[\mu].$$

Thus, by dominated convergence theorem $V_k u_k \to Vu$ in L^1_ρ. Setting $\zeta \in T(\Omega)$ and letting k tend to infinity in equality

$$\int_\Omega (-u_k \Delta\zeta + V_k u_k \zeta)\, dx = -\int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\mu, \tag{2.10}$$

implies that u satisfies (2.2). □

Remark. If V changes sign, we can put $\tilde{u} = u + \mathbb{K}[\mu]$. Then (2.1) is equivalent to

$$\left\{ \begin{array}{rll} -\Delta\tilde{u} + V\tilde{u} = V\mathbb{K}[\mu] & & \text{in } \Omega \\ \tilde{u} = 0 & & \text{in } \partial\Omega. \end{array} \right. \tag{2.11}$$

This is a Fredholm type problem (at least if the operator $\phi \mapsto R(v) := (-\Delta)^{-1}(V\phi)$ is compact in $L^1_\rho(\Omega)$). Existence will be ensured by orthogonality conditions.

If we assume that $V \geq 0$ and

$$\int_\Omega K^\Omega(x,y) V(x)\rho(x) dx < \infty, \tag{2.12}$$

for some $y \in \partial\Omega$, then δ_y is admissible. The following result yields to the solvability of (2.1) for any $\mu \in \mathfrak{M}_+(\Omega)$.

Proposition 2.5 *Assume $V \geq 0$ and the integrals* (2.12) *are bounded uniformly with respect to $y \in \partial\Omega$. Then any measure on $\partial\Omega$ is admissible.*

Proof. If M is the upper bound of these integrals and $\mu \in \mathfrak{M}_+(\partial\Omega)$, we have,

$$\int_\Omega \mathbb{K}[\mu](x) V(x)\rho(x) dx = \int_{\partial\Omega} \left(\int_\Omega K^\Omega(x,y) V(x)\rho(x) dx \right) d\mu(y) \leq M\mu(\partial\Omega), \tag{2.13}$$

by Fubini's theorem. Thus μ is admissible. □

Remark. Since the Poisson kernel in Ω satisfies the two-sided estimate

$$c^{-1} \frac{\rho(x)}{|x-y|^N} \leq K^\Omega(x,y) \leq c \frac{\rho(x)}{|x-y|^N} \qquad \forall (x,y) \in \Omega \times \partial\Omega, \tag{2.14}$$

for some $c > 0$, assumption (2.12) is equivalent to

$$\int_\Omega \frac{V(x)\rho^2(x)}{|x-y|^N} dx < \infty. \tag{2.15}$$

This implies (2.8) in particular. If we set $D_y = \max\{|x-y| : x \in \Omega\}$, then

$$\begin{aligned} \int_\Omega \frac{V(x)\rho^2(x)}{|x-y|^N} dx &= \int_0^{D_y} \left(\int_{\{x \in \Omega : |x-y| = r\}} V(x)\rho^2(x) dS_r(x) \right) \frac{dr}{r^N} \\ &= \lim_{\epsilon \to 0} \left(\left[r^{-N} \int_{\Omega \cap B_r(y)} V(x)\rho^2(x) dx \right]_\epsilon^{D_y} + N \int_\epsilon^{D_y} \left(\int_{\Omega \cap B_r(y)} V(x)\rho^2(x) dx \right) \frac{dr}{r^{N+1}} \right) \end{aligned}$$

(both quantity may be infinite). Thus, if we assume

$$\int_0^{D_y}\left(\int_{\Omega\cap B_r(y)} V(x)\rho^2(x)dx\right)\frac{dr}{r^{N+1}}<\infty, \tag{2.16}$$

there holds

$$\liminf_{\epsilon\to 0}\epsilon^{-N}\int_{\Omega\cap B_\epsilon(y)} V(x)\rho^2(x)dS=0. \tag{2.17}$$

Consequently

$$\int_\Omega \frac{V(x)\rho^2(x)}{|x-y|^N}dx=D_y^{-N}\int_\Omega V(x)\rho^2(x)dx+N\int_0^{D_y}\left(\int_{\Omega\cap B_r(y)} V(x)\rho^2(x)dx\right)\frac{dr}{r^{N+1}}. \tag{2.18}$$

Therefore (2.12) holds and δ_y is admissible.

As a natural extension of Proposition 2.5, we have the following stability result.

Theorem 2.6 *Assume $V\geq 0$ and*

$$\lim_{\substack{E\ Borel\\ |E|\to 0}}\int_E K^\Omega(x,y)V(x)\rho(x)dx=0 \quad \textit{uniformly with respect to } y\in\partial\Omega. \tag{2.19}$$

If μ_n is a sequence of positive Radon measures on $\partial\Omega$ converging to μ in the weak topology, then u_{μ_n} converges to u_μ in $L^1(\Omega)\cap L^1_{V\rho}(\Omega)$ and locally uniformly in Ω.*

Proof. We put $u_{\mu_n}:=u_n$. By the maximum principle $0\leq u_n\leq\mathbb{K}[\mu_n]$. Furthermore, it follows from (2.3) that

$$\|u_n\|_{L^1(\Omega)}+\|Vu_n\|_{L^1_\rho(\Omega)}\leq c\,\|\mu_n\|_{L^1(\partial\Omega)}\leq C. \tag{2.20}$$

Since $-\Delta u_n$ is bounded in $L^1_\rho(\Omega)$, the sequence $\{u_n\}$ is relatively compact in $L^1(\Omega)$ by the regularity theory for elliptic equations. Therefore, there exist a subsequence u_{n_k} and some function $u\in L^1(\Omega)$ with $Vu\in L^1_\rho(\Omega)$ such that u_{n_k} converges to u in $L^1(\Omega)$, almost everywhere on Ω and locally uniformly in Ω since $V\in L^\infty_{loc}(\Omega)$. The main question is to prove the convergence of Vu_{n_k} in $L^1_\rho(\Omega)$. If $E\subset\Omega$ is any Borel set, there holds

$$\begin{aligned}\int_E u_nV(x)\rho(x)dx&\leq\int_E\mathbb{K}[\mu_n]V(x)\rho(x)dx\\&\leq\int_{\partial\Omega}\left(\int_E K^\Omega(x,y)V(x)\rho(x)dx\right)d\mu_n(y)\\&\leq M_n\max_{y\in\partial\Omega}\int_E K^\Omega(x,y)V(x)\rho(x)dx,\end{aligned}$$

where $M_n:=\mu_n(\partial\Omega)$. Thus

$$\int_E u_nV(x)\rho(x)dx\leq M_n\max_{y\in\partial\Omega}\int_E K^\Omega(x,y)V(x)\rho(x)dx. \tag{2.21}$$

Then, by (2.19),

$$\lim_{|E|\to 0}\int_E u_n V(x)\rho(x)dx = 0.$$

As a consequence the set of function $\{u_n\rho V\}$ is uniformly integrable. By Vitali's theorem $Vu_{n_k}\to Vu$ in $L^1_\rho(\Omega)$. Since

$$\int_\Omega (-u_n\Delta\zeta + Vu_n\zeta)\,dx = -\int_{\partial\Omega}\frac{\partial\zeta}{\partial\mathbf{n}}d\mu_n, \tag{2.22}$$

for any $\zeta\in T(\Omega)$, the function u satisfies (2.2). □

Assumption (2.19) may be difficult to verify and the following result gives an easier formulation.

Proposition 2.7 *Assume $V\geq 0$ satisfies*

$$\lim_{\epsilon\to 0}\int_0^\epsilon\left(\int_{\Omega\cap B_r(y)}V(x)\rho^2(x)dx\right)\frac{dr}{r^{N+1}} = 0 \quad \textit{uniformly with respect to } y\in\partial\Omega. \tag{2.23}$$

Then (2.19) *holds.*

Proof. If $E\subset\Omega$ is a Borel set and $\delta>0$, we put $E_\delta = E\cap B_\delta(y)$ and $E^c_\delta = E\setminus E_\delta$. Then

$$\int_E\frac{V(x)\rho^2(x)}{|x-y|^N}dx = \int_{E_\delta}\frac{V(x)\rho^2(x)}{|x-y|^N}dx + \int_{E^c_\delta}\frac{V(x)\rho^2(x)}{|x-y|^N}dx.$$

Clearly

$$\int_{E^c_\delta}\frac{V(x)\rho^2(x)}{|x-y|^N}dx \leq \delta^{-N}\int_{E.}V(x)\rho^2(x)dx. \tag{2.24}$$

Since (2.16) holds for any $y\in\partial\Omega$, (2.18) implies

$$\int_{E_\delta}\frac{V(x)\rho^2(x)}{|x-y|^N}dx = \delta^{-N}\int_{E_\delta}V(x)\rho^2(x)dx + N\int_0^\delta\left(\int_{E\cap B_r(y)}V(x)\rho^2(x)dx\right)\frac{dr}{r^{N+1}}. \tag{2.25}$$

Using (2.23), for any $\epsilon>0$, there exists $s_0>0$ such that for any $s>0$ and $y\in\partial\Omega$

$$s\leq s_0 \Longrightarrow N\int_0^s\left(\int_{B_r(y)}V(x)\rho^2(x)dx\right)\frac{dr}{r^{N+1}}\leq\epsilon/2.$$

We fix $\delta = s_0$. Since (2.8) holds,

$$\lim_{\substack{E\text{ Borel}\\ |E|\,\to\,0}}\int_E V(x)\rho^2(x)dx = 0. \tag{2.26}$$

Then there exists $\eta>0$ such that for any Borel set $E\subset\Omega$,

$$|E|\leq\eta\Longrightarrow\int_E V(x)\rho^2(x)dx\leq s_0^N\epsilon/4.$$

Thus

$$\int_E \frac{V(x)\rho^2(x)}{|x-y|^N}dx \leq \epsilon.$$

This implies the claim by (2.14). $\square$

An assumption which is used in [18, Lemma 7.4] in order to prove the existence of a boundary trace of any positive solution of (1.22) is that there exists some nonnegative measurable function v defined on $\mathbb{R}_+$ such that

$$|V(x)| \leq v(\rho(x)) \quad \forall x \in \Omega \quad \text{and} \int_0^s tv(t)dt < \infty \quad \forall s > 0. \tag{2.27}$$

In the next result we show that condition (2.27) implies (2.19).

Proposition 2.8 *Assume V satisfies* (2.27). *Then*

$$\lim_{\substack{E \text{ Borel} \\ |E| \to 0}} \int_E K^\Omega(x,y)\,|V(x)|\,\rho(x)dx = 0 \quad \textit{uniformly with respect to } y \in \partial\Omega. \tag{2.28}$$

Proof. Since $\partial\Omega$ is C^2, there exist $\epsilon_0 > 0$ such that any for any $x \in \Omega$ satisfying $\rho(x) \leq \epsilon_0$, there exists a unique $\sigma(x) \in \partial\Omega$ such that $|x - \sigma(x)| = \rho(x)$. We use (2.23) in Proposition 2.7 under the equivalent form

$$\lim_{\epsilon \to 0} \int_0^\epsilon \left(\int_{\Omega \cap C_r(y)} |V(x)|\rho^2(x)dx \right) \frac{dr}{r^{N+1}} = 0 \quad \text{uniformly with respect to } y \in \partial\Omega, \tag{2.29}$$

in which we have replaced $B_r(y)$ by the the cylinder $C_r(y) := \{x \in \Omega : \rho(x) < r, |\sigma(x) - y| < r\}$. Then

$$\begin{aligned}\int_0^\epsilon \left(\int_{\Omega \cap C_r(y)} |V(x)|\rho^2(x)dx \right) \frac{dr}{r^{N+1}} &\leq c\int_0^\epsilon \left(\int_0^r v(t)t^2dt \right) \frac{dr}{r^2} \\ &\leq c\int_0^\epsilon v(t)\left(1 - \frac{t}{\epsilon}\right) tdt \\ &\leq c\int_0^\epsilon v(t)tdt.\end{aligned}$$

Thus (2.23) holds. $\square$

3 The capacitary approach

Throughout this section V is a locally bounded nonnegative and measurable function defined on Ω. We assume that there exists a positive measure μ_0 on $\partial\Omega$ such that

$$\int_\Omega \mathbb{K}[\mu_0]V(x)\rho(x)dx = \mathcal{E}(1,\mu_0) < \infty. \tag{3.1}$$

Definition 3.1 *If $\mu \in \mathfrak{M}_+(\partial\Omega)$ and f is a nonnegative measurable function defined in Ω such that*

$$(x,y) \mapsto \mathbb{K}[\mu](y)f(x)V(x)\rho(x) \in L^1(\Omega \times \partial\Omega; dx \otimes d\mu),$$

we set

$$\mathcal{E}(f,\mu) = \int_\Omega \left(\int_{\partial\Omega} K^\Omega(x,y)d\mu(y) \right) f(x)V(x)\rho(x)dx. \tag{3.2}$$

If we put

$$\check{\mathbb{K}}_V[f](y) = \int_\Omega K^\Omega(x,y)f(x)V(x)\rho(x)dx, \tag{3.3}$$

then, by Fubini's theorem, $\check{\mathbb{K}}_V[f] < \infty$, μ-almost everywhere on $\partial\Omega$ and

$$\mathcal{E}(f,\mu) = \int_{\partial\Omega} \left(\int_\Omega K^\Omega(x,y)f(x)V(x)\rho(x)dx \right) d\mu(y). \tag{3.4}$$

Proposition 3.2 *Let f be fixed. Then*
(a) $y \mapsto \check{\mathbb{K}}_V[f](y)$ is lower semicontinuous on $\partial\Omega$.
(b) $\mu \mapsto \mathcal{E}(f,\mu)$ is lower semicontinuous on $\mathfrak{M}_+(\partial\Omega)$ in the weak-topology*

Proof. Since $y \mapsto K^\Omega(x,y)$ is continuous, statement (a) follows by Fatou's lemma. If μ_n is a sequence in $\mathfrak{M}_+(\partial\Omega)$ converging to some μ in the weak*-topology, then $\mathbb{K}[\mu_n]$ converges to $\mathbb{K}[\mu]$ everywhere in Ω. By Fatou's lemma

$$\mathcal{E}(f,\mu) \leq \liminf_{n\to\infty} \int_\Omega \mathbb{K}[\mu_n](x)f(x)V(x)\rho(x)dx = \liminf_{n\to\infty} \mathcal{E}(f,\mu_n).$$

$\square$

Notice that if $V\rho f \in L^p(\Omega)$, for $p > N$, then $\mathbb{G}[Vf\rho] \in C^1(\overline{\Omega})$ and

$$\check{\mathbb{K}}[f](y) := \int_\Omega K^\Omega(x,y)V(x)f(x)\rho(x)dx = -\frac{\partial}{\partial \mathbf{n}}\mathbb{G}[Vf\rho](y). \tag{3.5}$$

This is in particular the case if f has compact support in Ω.

Definition 3.3 *We denote by $\mathfrak{M}^V(\partial\Omega)$ the set of all measures μ on $\partial\Omega$ such that $V\mathbb{K}[\mu] \in L^1_\rho(\Omega)$. If μ is such a measure, we denote*

$$\|\mu\|_{\mathfrak{M}^V} = \int_\Omega |\mathbb{K}[\mu](x)|\, V(x)\rho(x)dx = \|V\mathbb{K}[\mu]\|_{L^1_\rho}. \tag{3.6}$$

Clearly $\|\,.\,\|_{\mathfrak{M}^V}$ is a norm. The space $\mathfrak{M}^V(\partial\Omega)$ is not complete but its positive cone $\mathfrak{M}^V_+(\partial\Omega)$ is complete. If $E \subset \partial\Omega$ is a Borel subset, we put

$$\mathfrak{M}_+(E) = \{\mu \in \mathfrak{M}_+(\partial\Omega) : \mu(E^c) = 0\} \quad \text{and} \quad \mathfrak{M}^V_+(E) = \mathfrak{M}_+(E) \cap \mathfrak{M}^V(\partial\Omega).$$

Definition 3.4 *If $E \subset \partial\Omega$ is any Borel subset we set*

$$C_V(E) := \sup\{\mu(E) : \mu \in \mathfrak{M}_+^V(E), \|\mu\|_{\mathfrak{M}^V} \leq 1\}. \tag{3.7}$$

We notice that (3.7) is equivalent to

$$C_V(E) := \sup\left\{\frac{\mu(E)}{\|\mu\|_{\mathfrak{M}^V}} : \mu \in \mathfrak{M}_+^V(E)\right\}. \tag{3.8}$$

Proposition 3.5 *The set function C_V satisfies.*

$$C_V(E) \leq \sup_{y\in E}\left(\int_\Omega K^\Omega(x,y)V(x)\rho(x)dx\right)^{-1} \quad \forall E \subset \partial\Omega,\ E \text{ Borel}, \tag{3.9}$$

and equality holds in (3.9) *if E is compact. Moreover,*

$$C_V(E_1 \cup E_2) = \sup\{C_V(E_1), C_V(E_2)\} \quad \forall E_i \subset \partial\Omega,\ E_i \text{ Borel}. \tag{3.10}$$

Proof. Notice that $E \mapsto C_V(E)$ is a nondecreasing set function for the inclusion relation and that (3.7) implies

$$\mu(E) \leq C_V(E)\,\|\mu\|_{\mathfrak{M}^V} \qquad \forall \mu \in \mathfrak{M}_+^V(E). \tag{3.11}$$

Let $E \subset \partial\Omega$ be a Borel set and $\mu \in \mathfrak{M}_+(E)$. Then

$$\begin{aligned}\|\mu\|_{\mathfrak{M}^V} &= \int_E\left(\int_\Omega K^\Omega(x,y)V(x)\rho(x)dx\right)d\mu(y)\\ &\geq \mu(E)\inf_{y\in E}\int_\Omega K^\Omega(x,y)V(x)\rho(x)dx.\end{aligned}$$

Using (3.7) we derive

$$C_V(E) \leq \sup_{y\in E}\left(\int_\Omega K^\Omega(x,y)V(x)\rho(x)dx\right)^{-1}. \tag{3.12}$$

If E is compact, there exists $y_0 \in E$ such that

$$\inf_{y\in E}\int_\Omega K^\Omega(x,y)V(x)\rho(x)dx = \int_\Omega K^\Omega(x,y_0)V(x)\rho(x)dx,$$

since $y \mapsto \check{\mathbb{K}}[1](y)$ is l.s.c.. Thus

$$\|\delta_{y_0}\|_{\mathfrak{M}^V} = \delta_{y_0}(E)\int_\Omega K^\Omega(x,y_0)V(x)\rho(x)dx$$

and

$$C_V(E) \geq \frac{\delta_{y_0}(E)}{\|\delta_{y_0}\|_{\mathfrak{M}^V}} = \sup_{y\in E}\left(\int_\Omega K^\Omega(x,y)V(x)\rho(x)dx\right)^{-1}.$$

Therefore equality holds in (3.9). Identity (3.10) follows (3.9) when there is equality. Moreover it holds if E_1 and E_2 are two arbitrary compact sets. Since C_V is eventually an inner regular capacity (i.e. $C_V(E)=\sup\{C_V(K): K\subset E,\, K \text{ compact}\}$) it holds for any Borel set. However we give below a self-contained proof. If E_1 and E_2 be two disjoint Borel subsets of $\partial\Omega$, for any $\epsilon>0$ there exists $\mu\in\mathfrak{M}_+^V(E_1\cup E_2)$ such that

$$\frac{\mu(E_1)+\mu(E_2)}{\|\mu\|_{\mathfrak{M}^V}}\leq C_V(E_1\cup E_2)\leq\frac{\mu(E_1)+\mu(E_2)}{\|\mu\|_{\mathfrak{M}^V}}+\epsilon.$$

Set $\mu_i=\chi_{E_i}\mu$. Then $\mu_i\in\mathfrak{M}_+^V(E_i)$ and $\|\mu\|_{\mathfrak{M}^V}=\|\mu_1\|_{\mathfrak{M}^V}+\|\mu_2\|_{\mathfrak{M}^V}$. By (3.11)

$$C_V(E_1\cup E_2)\leq\frac{\|\mu_1\|_{\mathfrak{M}^V}}{\|\mu_1\|_{\mathfrak{M}^V}+\|\mu_2\|_{\mathfrak{M}^V}}C_V(E_1)+\frac{\|\mu_2\|_{\mathfrak{M}^V}}{\|\mu_1\|_{\mathfrak{M}^V}+\|\mu_2\|_{\mathfrak{M}^V}}C_V(E_2)+\epsilon \tag{3.13}$$

This implies that there exists $\theta\in[0,1]$ such that

$$C_V(E_1\cup E_2)\leq\theta C_V(E_1)+(1-\theta)C_V(E_2)\leq\max\{C_V(E_1),C_V(E_2)\}. \tag{3.14}$$

Since $C_V(E_1\cup E_2)\geq\max\{C_V(E_1),C_V(E_2)\}$ as C_V is increasing,

$$E_1\cap E_2=\emptyset\Longrightarrow C_V(E_1\cup E_2)=\max\{C_V(E_1),C_V(E_2)\}. \tag{3.15}$$

If $E_1\cap E_2\neq\emptyset$, then $E_1\cup E_2=E_1\cup(E_2\cap E_1^c)$ and therefore

$$C_V(E_1\cup E_2)=\max\{C_V(E_1),C_V(E_2\cap E_1^c)\}\leq\max\{C_V(E_1),C_V(E_2)\}.$$

Using again (3.8) we derive (3.10). □

The following set function is the dual expression of $C_V(E)$.

Definition 3.6 *For any Borel set $E\subset\partial\Omega$, we set*

$$C_V^*(E):=\inf\{\|f\|_{L^\infty}:\check{\mathbb{K}}[f](y)\geq 1\quad\forall y\in E\}. \tag{3.16}$$

The next result is stated in [11, p 922] using minimax theorem and the fact that K^Ω is lower semi continuous in $\Omega\times\partial\Omega$. Although the proof is not explicited, a simple adaptation of the proof of [1, Th 2.5.1] leads to the result.

Proposition 3.7 *For any compact set $E\subset\partial\Omega$,*

$$C_V(E)=C_V^*(E). \tag{3.17}$$

In the same paper [11], formula (3.9) with equality is claimed (if E is compact).

Theorem 3.8 *If $\{\mu_n\}$ is an increasing sequence of good measures converging to some measure μ in the weak* topology, then μ is good.*

Proof. We use formulation (4.10). We take for test function the function η solution of

$$\begin{cases} -\Delta\eta = 1 & \text{in } \Omega \\ \eta = 0 & \text{on } \Omega, \end{cases} \tag{3.18}$$

there holds

$$\int_\Omega (1+V)\, u_{\mu_n}\eta dx = -\int_{\partial\Omega} \frac{\partial\eta}{\partial\mathbf{n}} d\mu_n \leq c^{-1}\mu_n(\partial\Omega) \leq c^{-1}\mu(\partial\Omega)$$

where $c > 0$ is such that

$$c^{-1} \geq -\frac{\partial\eta}{\partial\mathbf{n}} \geq c \quad \text{on } \partial\Omega.$$

Since $\{u_{\mu_n}\}$ is increasing and $\eta \leq c\rho$ by Hopf boundary lemma, we can let $n \to \infty$ by the monotone convergence theorem. If $u := \lim_{n\to\infty} u_{\mu_n}$, we obtain

$$\int_\Omega (1+V)\, u\eta dx \leq c^{-1}\mu(\partial\Omega).$$

Thus u and $\rho V u$ are in $L^1(\Omega)$. Next, if $\zeta \in C_0^1(\overline{\Omega}) \cap C^{1,1}(\overline{\Omega})$, then $u_{\mu_n}|\Delta\zeta| \leq C u_{\mu_n}$ and $V u_{\mu_n}|\zeta| \leq C V u_{\mu_n}\eta$. Because the sequence $\{u_{\mu_n}\}$ and $\{V u_{\mu_n}\eta\}$ are uniformly integrable, the same holds for $\{u_{\mu_n}\Delta\zeta\}$ and $\{V u_{\mu_n}\zeta\}$. Considering

$$\int_\Omega (-u_{\mu_n}\Delta\zeta + V u_{\mu_n}\zeta)\, dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu_n.$$

it follows by Vitali's theorem,

$$\int_\Omega (-u\Delta\zeta + V u\zeta)\, dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu.$$

Thus μ is a good measure. $\square$

We define the *singular boundary set* Z_V by

$$Z_V = \left\{ y \in \partial\Omega : \int_\Omega K^\Omega(x,y)V(x)\rho(x)dx = \infty \right\}. \tag{3.19}$$

Since $\check{\mathbb{K}}[1]$ is l.s.c., it is a Borel function and Z_V is a Borel set. The next result characterizes the good measures.

Proposition 3.9 *Let μ be an admissible positive measure. Then $\mu(Z_V) = 0$.*

Proof. If $K \subset Z_V$ is compact, $\mu_K = \chi_K\mu$ is admissible, thus, by Fubini theorem

$$\|\mu_K\|_{\mathfrak{M}^V} = \int_K \left(\int_\Omega K^\Omega(x,y)V(x)\rho(x)dx \right) d\mu(y) < \infty.$$

Since

$$\int_\Omega K^\Omega(x,y)V(x)\rho(x)dx \equiv \infty \qquad \forall y \in K$$

it follows that $\mu(K) = 0$. This implies $\mu(Z_V) = 0$ by regularity. $\square$

Theorem 3.10 *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$ such that*

$$\mu(Z_V) = 0. \tag{3.20}$$

Then μ is good.

Proof. Since $\check{\mathbb{K}}[1]$ is l.s.c., for any $n \in \mathbb{N}_*$,

$$K_n := \{y \in \partial\Omega : \check{\mathbb{K}}[1](y) \leq n\}$$

is a compact subset of $\partial\Omega$. Furthermore $K_n \cap Z_V = \emptyset$ and $\cup K_n = Z_V^c$. Let $\mu_n = \chi_{K_n}\mu$, then

$$\mathcal{E}(1, \mu_n) = \int_\Omega \mathbb{K}[\mu_n]V(x)\rho(x)dx \leq n\mu_n(K_n). \tag{3.21}$$

Therefore μ_n is admissible. By the monotone convergence theorem, $\mu_n \uparrow \chi_{Z_V c}\mu$ and by Theorem 3.8, $\chi_{Z_V c}\mu$ is good. Since (5.7) holds, $\chi_{Z_V c}\mu = \mu$, which ends the proof. □

The full characterization of the good measures in the general case appears to be difficult without any further assumptions on V. However the following holds

Theorem 3.11 *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$ be a good measure. The following assertions are equivalent:*
(i) $\mu(Z_V) = 0$.
(ii) There exists an increasing sequence of admissible measures $\{\mu_n\}$ which converges to μ in the weak-topology.*

Proof. If (i) holds, it follows from the proof of Theorem 3.10 that the sequence $\{\mu_n\}$ increases and converges to μ. If (ii) holds, any admissible measure μ_n vanishes on Z_V by Proposition 3.9. Since $\mu_n \leq \mu$, there exists an increasing sequence of μ-integrable functions h_n such that $\mu_n = h_n\mu$. Then $\mu_n(Z_V)$ increases to $\mu(Z_V)$ by the monotone convergence theorem. The conclusion follows from the fact that $\mu_n(Z_V) = 0$. □

4 Representation formula and reduced measures

We recall the construction of the Poisson kernel for $-\Delta + V$: if we look for a solution of

$$\left\{\begin{array}{ll} -\Delta v + V(x)v = 0 & \text{in } \Omega \\ v = \nu & \text{in } \partial\Omega, \end{array}\right. \tag{4.1}$$

where $\nu \in \mathfrak{M}(\partial\Omega)$, $V \geq 0$, $V \in L^\infty_{loc}(\Omega)$, we can consider an increasing sequence of smooth domains Ω_n such that $\overline{\Omega}_n \subset \Omega_{n+1}$ and $\cup_n\Omega_n = \cup_n\overline{\Omega}_n = \Omega$. For each of these domains, denote by $K^\Omega_{V\chi_{\Omega_n}}$ the Poisson kernel of $-\Delta + V\chi_{\Omega_n}$ in Ω and by $\mathbb{K}_{V\chi_{\Omega_n}}[.]$ the corresponding operator. We denote by $K^\Omega := K^\Omega_0$ the Poisson kernel in Ω and by $\mathbb{K}[.]$ the Poisson operator in Ω. Then the solution $v := v_n$ of

$$\left\{\begin{array}{ll} -\Delta v + V\chi_{\Omega_n}v = 0 & \text{in } \Omega \\ v = \nu & \text{in } \partial\Omega, \end{array}\right. \tag{4.2}$$

is expressed by

$$v_n(x) = \int_{\partial\Omega} K^{\Omega}_{V\chi_{\Omega_n}}(x,y)d\nu(y) = \mathbb{K}_{V\chi_{\Omega_n}}[\nu](x). \tag{4.3}$$

If G^{Ω} is the Green kernel of $-\Delta$ in Ω and $\mathbb{G}[.]$ the corresponding Green operator, (4.3) is equivalent to

$$v_n(x) + \int_{\Omega} G^{\Omega}(x,y)(V\chi_{\Omega_n} v_n)(y)dy = \int_{\partial\Omega} K^{\Omega}(x,y)d\nu(y), \tag{4.4}$$

equivalently

$$v_n + \mathbb{G}[V\chi_{\Omega_n} v_n] = \mathbb{K}[\nu].$$

Notice that this equality is equivalent to the weak formulation of problem (4.2): for any $\zeta \in T(\Omega)$, there holds

$$\int_{\Omega} \left(-v_n\Delta\zeta + V\chi_{\Omega_n} v_n\zeta\right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\nu. \tag{4.5}$$

Since $n \mapsto K^{\Omega}_{V\chi_{\Omega_n}}$ is decreasing, the sequence $\{v_n\}$ inherits this property and there exists

$$\lim_{n\to\infty} K^{\Omega}_{V\chi_{\Omega_n}}(x,y) = K^{\Omega}_{V}(x,y). \tag{4.6}$$

By the monotone convergence theorem,

$$\lim_{n\to\infty} v_n(x) = v(x) = \int_{\partial\Omega} K^{\Omega}_{V}(x,y)d\nu(y). \tag{4.7}$$

By Fatou's theorem

$$\int_{\Omega} G^{\Omega}(x,y)V(y)v(y)dy \le \liminf_{n\to\infty} \int_{\Omega} G^{\Omega}(x,y)(V\chi_{\Omega_n} v_n)(y)dy, \tag{4.8}$$

and thus,

$$v(x) + \int_{\Omega} G^{\Omega}(x,y)V(y)v(y)dy \le \mathbb{K}[\nu](x) \qquad \forall x \in \Omega. \tag{4.9}$$

Now the main question is to know whether v keeps the boundary value ν. Equivalently, whether the equality holds in (4.8) with lim instead of lim inf, and therefore in (4.9). This question is associated to the notion of reduced measured in the sense of Brezis-Marcus-Ponce: Since $Vv \in L^1_\rho(\Omega)$ and

$$-\Delta v + V(x)v = 0 \qquad \text{in } \Omega \tag{4.10}$$

holds, the function $v + \mathbb{G}[Vv]$ is positive and harmonic in Ω. Thus it admits a boundary trace $\nu^* \in \mathfrak{M}_+(\partial\Omega)$ and

$$v + \mathbb{G}[Vv] = \mathbb{K}[\nu^*]. \tag{4.11}$$

Equivalently v satisfies the relaxed problem

$$\begin{cases} -\Delta v + V(x)v = 0 & \text{in } \Omega \\ \qquad\qquad\quad v = \nu^* & \text{in } \partial\Omega, \end{cases} \tag{4.12}$$

and thus $v = u_{\nu^*}$. Noticed that $\nu^* \le \nu$ and the mapping $\nu \mapsto \nu^*$ is nondecreasing.

Definition 4.1 *The measure ν^* is the reduced measure associated to ν.*

Proposition 4.2 *There holds $\mathbb{K}_V[\nu] = \mathbb{K}_V[\nu^*]$. Furthermore the reduced measure ν^* is the largest measure for which the following problem*

$$\left\{\begin{aligned} -\Delta v + V(x)v = 0 \qquad & in\ \Omega \\ \lambda \in \mathfrak{M}_+(\partial\Omega),\ \lambda \leq \nu \qquad & \\ v = \lambda \qquad & in\ \partial\Omega, \end{aligned}\right. \tag{4.13}$$

admits a solution.

Proof. The first assertion follows from the fact that $v = \mathbb{K}_V[\nu]$ by (4.6) and $v = u_{\nu^*} = \mathbb{K}_V[\nu^*]$ by (4.12). It is clear that $\nu^* \leq \nu$ and that the problem (4.13) admits a solution for $\lambda = \nu^*$. If λ is a positive measure smaller than μ, then $\lambda^* \leq \mu^*$. But if there exist some λ such that the problem (4.13) admits a solution, then $\lambda = \lambda^*$. This implies the claim. □

As a consequence of the characterization of ν^* there holds

Corollary 4.3 *Assume $V \geq 0$ and let $\{V_k\}$ be an increasing sequence of nonnegative bounded measurable functions converging to V a.e. in Ω. Then the solution u_k of*

$$\left\{\begin{aligned} -\Delta u + V_k u = 0 \qquad & in\ \Omega \\ u = \nu \qquad & in\ \partial\Omega, \end{aligned}\right. \tag{4.14}$$

converges to u_{ν^*}.

Proof. The previous construction shows that $u_k = \mathbb{K}_{V_k}[\nu]$ decreases to some $\tilde{u}$ which satisfies a relaxed equation, the boundary data of which, $\tilde{\nu}^*$, is the largest measure $\lambda \leq \nu$ for which problem (4.13) admits a solution. Therefore $\tilde{\nu}^* = \nu^*$ and $\tilde{u} = u_{\nu^*}$. Similarly $\{K^{\Omega}_{V_k}\}$ decreases and converges to K^{Ω}_V. □

We define the boundary vanishing set of K^{Ω}_V by

$$Z^*_V := \{y \in \partial\Omega \,|\, K^{\Omega}_V(x,y) = 0\} \quad \text{for some } x \in \Omega. \tag{4.15}$$

Since $V \in L^\infty_{loc}(\Omega)$, Z^*_V is independent of x by Harnack inequality; furthermore it is a Borel set.

Theorem 4.4 *Let $\nu \in \mathfrak{M}_+(\partial\Omega)$.*
(i) If $\nu((Z^*_V)^c) = 0$, then $\nu^* = 0$.
(ii) There always holds $Z^*_V \subset Z_V$.

Proof. The first assertion is clear since $\nu = \chi_{Z^*_V}\nu + \chi_{(Z^*_V)^c}\nu = \chi_{Z^*_V}\nu$ and, by Proposition 4.2,

$$u_{\nu^*}(x) = \mathbb{K}_V[\nu^*](x) = \int_{Z^*_V} K^{\Omega}_V(x,y)d\nu(y) = 0 \qquad \forall x \in \Omega,$$

by definition of Z_V^*. For proving (ii), we assume that $C_V(Z_V^*) > 0$; there exists $\mu \in \mathfrak{M}_+^V(Z_V^*)$ such that $\mu(Z_V^*) > 0$. Since μ is admissible let u_μ be the solution of (1.1). Then $\mu^* = \mu$, thus $u_\mu = \mathbb{K}^V[\mu]$ and

$$\mathbb{K}^V[\mu](x) = \int_{\partial\Omega} K_V^\Omega(x,y)d\mu(y) = \int_{Z_V^*} K_V^\Omega(x,y)d\mu(y) = 0,$$

contradiction. Thus $C_V(Z_V^*) = 0$. Since (3.9) implies that Z_V is the largest Borel set with zero C_V-capacity, it implies $Z_V^* \subset Z_V$. □

In order to obtain more precise informations on Z_V^* some minimal regularity assumptions on V are needed. We also recall the following result proved by Ancona [2].

Theorem 4.5 *Assume $V \geq 0$ satisfies $\rho^2 V \in L^\infty(\Omega)$. If for some $y_0 \in \partial\Omega$ and any cone C_{y_0} with vertex y_0 having the property that $\overline{C}_{y_0} \cap B_r(y_0) \subset \Omega \cup \{y_0\}$ for some $r > 0$, there exists $c_1 > 0$ such that*

$$\forall (x,y) \in \Omega \cap B_r(y_0) \times \Omega \cap B_r(y_0),\ |x - y_0| = |y - y_0| \leq r \Longrightarrow c^{-1} \leq \frac{V(x)}{V(y)} \leq c_1 \tag{4.16}$$

and

$$\int_0^r V(t\mathbf{n}_{y_0})tdt = \infty, \tag{4.17}$$

where $\mathbf{n_0}$ is the normal outward unit vector to $\partial\Omega$ at y_0, then

$$K_V^\Omega(x,y_0) = 0 \qquad \forall x \in \Omega. \tag{4.18}$$

We define the *conical singular boundary set*

$$\tilde{Z}_V = \left\{ y \in \partial\Omega : \int_{\Omega\cap C_y} K^\Omega(x,y)V(x)\rho(x)dx = \infty \text{ for some cone } C_y \Subset \Omega \right\} \tag{4.19}$$

where $C_y \Subset \Omega$ means that there exists $a > 0$ such that $\overline{C}_y \cap B_a(y) \subset \Omega \cup \{y\}$. Clearly $\tilde{Z}_V \subset Z_V$.

Corollary 4.6 *Assume $V \geq 0$ satisfies $\rho^2 V \in L^\infty(\Omega)$ and the conical oscillation condition (4.16) of Theorem 4.5 for any $y \in Z_V$. Then $\tilde{Z}_V = Z_V^*$.*

Proof. We can assume that $y = 0$ and denote $C_y = C$. Since

$$K^\Omega(x,0)V(x)\rho(x) \leq ca^{-N}V(x)\rho^2(x) \qquad \forall x \in \Omega \cap B_a^c,$$

and $V\rho^2 \in L^1(\Omega)$, there holds, using (2.14),

$$\int_{B_a\cap C} V(x)\rho^2(x)\frac{dx}{|x|^N} = \infty.$$

Using spherical coordinates and the fact that $\rho^2(x) \geq c|x|$ in $B_a \cap C_y$,

$$\int_0^a \int_S V(r,\sigma)rd\sigma\, dr = \infty.$$

where $S = C \cap \partial B_1$. But in $C \cap B_a$ the oscillation condition (4.16) holds. This implies

$$\int_0^a V(r,\sigma)tdt = \infty \qquad \forall \sigma \in S. \tag{4.20}$$

Thus $y \in Z_V^*$. □

5 The boundary trace

5.1 The regular part

In this section, $V \in L^{\infty}_{loc}(\Omega)$ is nonnegative. If $0 < \epsilon \leq \epsilon_0$, we denote $d(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$, and set $\Omega_\epsilon := \{x \in \Omega : d(x) > \epsilon\}$, $\Omega'_\epsilon = \Omega \setminus \Omega_\epsilon$ and $\Sigma_\epsilon = \partial\Omega_\epsilon$. It is well known that there exists ϵ_0 such that, for any $0 < \epsilon \leq \epsilon_0$ and any $x \in \Omega'_\epsilon$ there exists a unique projection $\sigma(x)$ of x on $\partial\Omega$ and any $x \in \Omega'_\epsilon$ can be written in a unique way under the form

$$x = \sigma(x) - d(x)\mathbf{n}$$

where $\mathbf{n}$ is the outward normal unit vector to $\partial\Omega$ at $\sigma(x)$. The mapping $x \mapsto (d(x), \sigma(x))$ is a C^2 diffeomorphism from Ω'_ϵ to $(0, \epsilon_0] \times \partial\Omega$. We recall the following definition given in [18]. If $\mathcal{A}$ is a Borel subset of $\partial\Omega$, we set $\mathcal{A}_\epsilon = \{x \in \Sigma_\epsilon : \sigma(x) \in A\}$.

Definition 5.1 *Let $\mathcal{A}$ be a relatively open subset of $\partial\Omega$, $\{\mu_\epsilon\}$ be a set of Radon measures on $\mathcal{A}_\epsilon$ $(0 < \epsilon \leq \epsilon_0)$ and $\mu \in \mathfrak{M}(\mathcal{A})$. We say that $\mu_\epsilon \rightharpoonup \mu$ in the weak*-topology if, for any $\zeta \in C_c(\mathcal{A})$,*

$$\lim_{\epsilon \to 0} \int_{\mathcal{A}_\epsilon} \zeta(\sigma(x)) d\mu_\epsilon(x) = \int_{\mathcal{A}} \zeta d\mu. \tag{5.1}$$

A function $u \in C(\Omega)$ possesses a boundary trace $\mu \in \mathfrak{M}(\mathcal{A})$ if

$$\lim_{\epsilon \to 0} \int_{\mathcal{A}_\epsilon} \zeta(\sigma(x)) u(x) dS(x) = \int_{\mathcal{A}} \zeta d\mu \qquad \forall \zeta \in C_c(\mathcal{A}). \tag{5.2}$$

The following result is proved in [18, p 694].

Proposition 5.2 *Let $u \in C(\Omega)$ be a positive solution of*

$$-\Delta u + V(x)u = 0 \qquad \text{in } \Omega. \tag{5.3}$$

Assume that, for some $z \in \partial\Omega$, there exists an open neighborhood U of z such that

$$\int_{U \cap \Omega} V u \rho(x) dx < \infty. \tag{5.4}$$

Then $u \in L^1(K \cap \Omega)$ for any compact subset $K \subset G$ and there exists a positive Radon measure μ on $\mathcal{A} = U \cap \partial\Omega$ such that

$$\lim_{\epsilon \to 0} \int_{U \cap \Sigma_\epsilon} \zeta(\sigma(x)) u(x) dS(x) = \int_{\mathcal{A}} \zeta d\mu \qquad \forall \zeta \in C_c(U \cap \Omega). \tag{5.5}$$

Notice that any continuous solution of (5.3) in Ω belongs to $W^{2,p}_{loc}(\Omega)$ for any $(1 \leq p < \infty)$. This previous result yields to a natural definition of the regular boundary points.

Definition 5.3 *Let $u \in C(\Omega)$ be a positive solution of* (5.3). *A point $z \in \partial\Omega$ is called a regular boundary point for u if there exists an open neighborhood U of z such that* (5.31) *holds. The set of regular boundary points is a relatively open subset of $\partial\Omega$, denoted by $\mathcal{R}(u)$. The set $\mathcal{S}(u) = \partial\Omega \setminus \mathcal{R}(u)$ is the singular boundary set of u. It is a closed set.*

By Proposition 5.2 and using a partition of unity, we see that there exists a positive Radon measure $\mu := \mu_u$ on $\mathcal{R}(u)$ such that (5.5) holds with U replaced by $\mathcal{R}(u)$. The couple $(\mu_u, \mathcal{S}(u))$ is called the **boundary trace of** u. *The main question of the boundary trace problem is to analyse the behaviour of* u *near the set* $\mathcal{S}(u)$.

For any positive good measure μ on $\partial\Omega$, we denote by u_μ the solution of (4.1) defined by (4.10)-(4.11).

Proposition 5.4 *Let* $u \in C(\Omega) \cap W^{2,p}_{loc}(\Omega)$ *for any* $(1 \leq p < \infty)$ *be a positive solution of* (5.3) *in* Ω *with boundary trace* $(\mu_u, \mathcal{S}(u))$. *Then* $u \geq u_{\mu_u}$.

Proof. Let $G \subset \partial\Omega$ be a relatively open subset such that $\overline{G} \subset \mathcal{R}(u)$ with a C^2 relative boundary $\partial^* G = \overline{G} \setminus G$. There exists an increasing sequence of C^2 domains Ω_n such that $\overline{G} \subset \partial\Omega_n$, $\partial\Omega_n \setminus \overline{G} \subset \Omega$ and $\cup_n \Omega_n = \Omega$. For any n, let $v := v_n$ be the solution of

$$\left\{ \begin{array}{ll} -\Delta v + V v = 0 & \text{in } \Omega_n \\ v = \chi_G \mu & \text{in } \partial\Omega_n. \end{array} \right. \tag{5.6}$$

Let u_n be the restriction of u to Ω_n. Since $u \in C(\Omega)$ and $V u \rho \in L^1(\Omega_n)$, there also holds $V u \rho_n \in L^1(\Omega_n)$ where we have denoted by ρ_n the first eigenfunction of $-\Delta$ in $W^{1,2}_0(\Omega_n)$. Consequently u_n admits a regular boundary trace μ_n on $\partial\Omega_n$ (i.e. $\mathcal{R}(u_n) = \partial\Omega_n$) and u_n is the solution of

$$\left\{ \begin{array}{ll} -\Delta v + V v = 0 & \text{in } \Omega_n \\ v = \mu_n & \text{in } \partial\Omega_n. \end{array} \right. \tag{5.7}$$

Furthermore $\mu_n|_G = \chi_G \mu_u$. It follows from Brezis estimates and in particular (2.5) that $u_n \leq u$ in Ω_n. Since $\Omega_n \subset \Omega_{n+1}$, $v_n \leq v_{n+1}$. Moreover

$$v_n + \mathbb{G}^{\Omega_n}[V v_n] = \mathbb{K}^{\Omega_n}[\chi_G \mu] \qquad \text{in } \Omega_n.$$

Since $\mathbb{K}^{\Omega_n}[\chi_G \mu_u] \to \mathbb{K}^{\Omega}[\chi_G \mu_u]$, and the Green kernels $G^{\Omega_n}(x, y)$ are increasing with n, it follows from monotone convergence that $v_n \uparrow v$ and there holds

$$v + \mathbb{G}^{\Omega}[V v] = \mathbb{K}^{\Omega}[\chi_G \mu_u] \qquad \text{in } \Omega.$$

Thus $v = u_{\chi_G \mu_u}$ and $u_{\chi_G \mu_u} \leq u$. We can now replace G by a sequence $\{G_k\}$ of relatively open sets with the same properties as G, $\overline{G}_k \subset G_k$ and $\cup_k G_k = \mathcal{R}(u)$. Then $\{u_{\chi_{G_k} \mu_u}\}$ is increasing and converges to some $\tilde{u}$. Since

$$u_{\chi_{Gk} \mu_u} + \mathbb{G}^{\Omega}[V u_{\chi_{Gk} \mu_u}] = \mathbb{K}^{\Omega}[\chi_{Gk} \mu_u],$$

and $\mathbb{K}^{\Omega}[\chi_{Gk} \mu] \uparrow \mathbb{K}^{\Omega}[\mu_u]$, we derive

$$\tilde{u} + \mathbb{G}^{\Omega}[V \tilde{u}] = \mathbb{K}^{\Omega}[\mu_u].$$

This implies that $\tilde{u} = u_{\mu_u} \leq u$. $\square$

5.2 The singular part

The following result is essentially proved in [18, Lemma 2.8].

Proposition 5.5 *Let $u \in C(\Omega)$ for any $(1 \leq p < \infty)$ be a positive solution of (5.3) and suppose that $z \in \mathcal{S}(u)$ and that there exists an open neighborhood U_0 of z such that $u \in L^1(\Omega \cap U_0)$. Then for any open neighborhood U of z, there holds*

$$\lim_{\epsilon \to 0} \int_{U \cap \Sigma_\epsilon} \zeta(\sigma(x)) u(x) dS(x) = \infty. \tag{5.8}$$

As immediate consequences, we have

Corollary 5.6 *Assume u satisfies the regularity assumption of Proposition 5.4. Then for any $z \in \mathcal{S}(u)$ and any open neighborhood U of z, there holds*

$$\limsup_{\epsilon \to 0} \int_{U \cap \Sigma_\epsilon} \zeta(\sigma(x)) u(x) dS(x) = \infty. \tag{5.9}$$

Corollary 5.7 *Assume u satisfies the regularity assumption of Proposition 5.4. If $u \in L^1(\Omega)$, Then for any $z \in \mathcal{S}(u)$ and any open neighborhood U of z, (5.8) holds.*

The two next results give conditions on V which imply that $\mathcal{S}(u) = \emptyset$.

Theorem 5.8 *Assume $N = 2$, V is nonnegative and satisfies (2.19). If u is a positive solution of (5.3), then $\mathcal{R}(u) = \partial\Omega$.*

Proof. We assume that

$$\int_\Omega V \rho u dx = \infty. \tag{5.10}$$

If $0 < \epsilon \leq \epsilon_0$, we denote by $(\rho_\epsilon, \lambda_\epsilon)$ are the normalized first eigenfunction and first eigenvalue of $-\Delta$ in $W_0^{1,2}(\Omega_\epsilon)$, then

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} V \rho_\epsilon u dx = \infty. \tag{5.11}$$

Because

$$\int_{\Omega_\epsilon} (\lambda_\epsilon + \rho_\epsilon V) u dx = -\int_{\partial\Omega_\epsilon} \frac{\partial \rho_\epsilon}{\partial \mathbf{n}} u dS,$$

and

$$c^{-1} \leq -\frac{\partial \rho_\epsilon}{\partial \mathbf{n}} \leq c,$$

for some $c > 1$ independent of ϵ, there holds

$$\lim_{\epsilon \to 0} \int_{\partial\Omega_\epsilon} u dS = \infty. \tag{5.12}$$

Denote by m_ϵ this last integral and set $v_\epsilon = m_\epsilon^{-1} u$ and $\mu_\epsilon = m_\epsilon^{-1} u|_{\partial\Omega_\epsilon}$. Then

$$v_\epsilon + \mathbb{G}^{\Omega_\epsilon}[V v_\epsilon] = \mathbb{K}^{\Omega_\epsilon}[\mu_\epsilon] \qquad \text{in } \Omega_\epsilon \tag{5.13}$$

where

$$\mathbb{K}^{\Omega_\epsilon}[\mu_\epsilon](x) = \int_{\partial\Omega_\epsilon} K^{\Omega_\epsilon}(x,y)\mu_\epsilon(y)dS(y) \tag{5.14}$$

is the Poisson potential of μ_ϵ in Ω_ϵ and

$$\mathbb{G}^{\Omega_\epsilon}[Vu](x) = \int_{\Omega_\epsilon} G^{\Omega_\epsilon}(x,y)V(y)u(y)dy,$$

the Green potential of Vu in Ω_ϵ. Furthermore

$$\left\{ \begin{aligned} -\Delta v_\epsilon + Vv_\epsilon &= 0 \qquad && \text{in } \Omega_\epsilon \\ v_\epsilon &= \mu_\epsilon \qquad && \text{in } \partial\Omega_\epsilon. \end{aligned} \right. \tag{5.15}$$

By Brezis estimates and regularity theory for elliptic equations, $\{\chi_{\Omega_\epsilon} v_\epsilon\}$ is relatively compact in $L^1(\Omega)$ and in the local uniform topology of Ω_ϵ. Up to a subsequence $\{\epsilon_n\}$, μ_{ϵ_n} converges to a probability measure μ on $\partial\Omega$ in the weak*-topology. It is classical that

$$\mathbb{K}^{\Omega_{\epsilon_n}}[\mu_{\epsilon_n}] \to \mathbb{K}[\mu]$$

locally uniformly in Ω, and $\chi_{\Omega_{\epsilon_n}} v_{\epsilon_n} \to v$ in the local uniform topology of Ω, and a.e. in Ω. Because $G^{\Omega_\epsilon}(x,y) \uparrow G^{\Omega}(x,y)$, there holds for any $x \in \Omega$

$$\lim_{n\to\infty} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y)V(y)v_{\epsilon_n}(y) = G^{\Omega}(x,y)V(y)v(y) \quad \text{for almost all } y \in \Omega \tag{5.16}$$

Furthermore $v_{\epsilon_n} \le \mathbb{K}^{\Omega_{\epsilon_n}}[\mu_{\epsilon_n}]$ reads

$$v_{\epsilon_n}(y) \le c\rho_{\epsilon_n}(y) \int_{\partial\Omega_n} \frac{\mu_{\epsilon_n}(z)dS(z)}{|y-z|^2}.$$

In order to go to the limit in the expression

$$L_n := \mathbb{G}^{\Omega_{\epsilon_n}}[Vv_{\epsilon_n}](x) = \int_{\Omega} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y)V(y)v_{\epsilon_n}(y)dy, \tag{5.17}$$

we may assume that $x \in \Omega_{\epsilon_1}$ where $0 < \epsilon_1 \le \epsilon_0$ is fixed and write $\Omega = \Omega_{\epsilon_1} \cup \Omega'_{\epsilon_1}$ where

$$\Omega'_{\epsilon_1} = \Omega \setminus \Omega_{\epsilon_1} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \epsilon_1\}$$

and $L_n = M_n + P_n$ where

$$M_n = \int_{\Omega_{\epsilon_1}} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y)V(y)v_{\epsilon_n}(y)dy \tag{5.18}$$

and

$$P_n = \int_{\Omega'_{\epsilon_1}} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y)V(y)v_{\epsilon_n}(y)dy. \tag{5.19}$$

Since

$$\begin{aligned} \chi_{\Omega_{\epsilon_1}}(y) G^{\Omega_{\epsilon_n}}(x,y)V(y)v_{\epsilon_n}(y) &\le c\chi_{\Omega_{\epsilon_1}}(y) \left|\ln(|x-y|)\right| V(y)v_{\epsilon_n}(y) \\ &\le c\,\|V\|_{L^\infty(\Omega_{\epsilon_1})} \chi_{\Omega_{\epsilon_1}}(y) \left|\ln(|x-y|)\right| v_{\epsilon_n}(y), \end{aligned}$$

it follows by the dominated convergence theorem that

$$\lim_{n\to\infty} M_n = \int_{\Omega_{\epsilon_1}} G^{\Omega}(x,y)V(y)v(y)dy. \tag{5.20}$$

Let $E \subset \Omega$ be a Borel subset. Then $G^{\Omega_{\epsilon_n}}(x,y) \leq c(x)\rho_{\epsilon_n}(y)$ if $y \in \Omega'_{\epsilon_1}$. By Fubini,

$$\begin{aligned}\int_{\Omega'_{\epsilon_1}\cap E} \chi_{\Omega_{\epsilon_n}}(y)G^{\Omega_{\epsilon_n}}(x,y)V(y)v_{\epsilon_n}(y)dy &\leq cc(x)\int_{\partial\Omega_n}\left(\int_{\Omega'_{\epsilon_1}\cap E}\chi_{\Omega_{\epsilon_n}}(y)\frac{\rho^2_{\epsilon_n}(y)V(y)}{|y-z|^2}dy\right)\mu_{\epsilon_n}(z)dS(z)\\ &\leq cc(x)\max_{z\in\partial\Omega_{\epsilon_n}}\int_{\Omega'_{\epsilon_1}\cap E}\chi_{\Omega_{\epsilon_n}}(y)\frac{\rho^2_{\epsilon_n}(y)V(y)}{|y-z|^2}dy\end{aligned} \tag{5.21}$$

If $y \in \Omega_{\epsilon_n} \cap E$, there holds $\rho(y) = \rho_{\epsilon_n}(y) + \epsilon_n$. If $z \in \partial\Omega_{\epsilon_n} \cap E$ and we denote by $\sigma(z)$ the projection of z onto $\partial\Omega$, there holds $|y - \sigma(z)| \leq |y - z| + \epsilon_n$. By monotonicity

$$\frac{\rho_{\epsilon_n}(y)}{|y-z|} \leq \frac{\rho_{\epsilon_n}(y)+\epsilon_n}{|y-z|+\epsilon_n} \leq \frac{\rho(y)}{|y-\sigma(z)|}, \tag{5.22}$$

thus

$$\int_{\Omega'_{\epsilon_1}\cap E} \chi_{\Omega_{\epsilon_n}}(y)G^{\Omega_{\epsilon_n}}(x,y)V(y)v_{\epsilon_n}(y)dy \leq cc(x)\max_{z\in\partial\Omega}\int_{\Omega'_{\epsilon_1}\cap E}\chi_{\Omega_{\epsilon_n}}(y)\frac{\rho^2(y)V(y)}{|y-z|^2}dy. \tag{5.23}$$

By (2.19) this last integral goes to zero if $\left|\Omega'_{\epsilon_1} \cap E \cap \Omega_{\epsilon_n}\right| \to 0$. Thus by Vitali's theorem, the sequence of functions $\{\chi_{\Omega_{\epsilon_n}}(.)G^{\Omega_{\epsilon_n}}(x,.)V(y)v_{\epsilon_n}(.)\}_{n\in\mathbb{N}}$ is uniformly integrable in y, for any $x \in \Omega$. It implies that

$$\lim_{n\to\infty}\int_{\Omega}\chi_{\Omega_{\epsilon_n}}(y)G^{\Omega_{\epsilon_n}}(x,y)V(y)v_{\epsilon_n}(y)dy = \int_{\Omega}G^{\Omega}(x,y)V(y)v(y)dy, \tag{5.24}$$

and there holds $v + \mathbb{G}[Vv] = \mathbb{K}[\mu]$. Since $u = m_\epsilon v_\epsilon$ in Ω and $m_\epsilon \to \infty$, we get a contradiction since it would imply $u \equiv \infty$. □

In order to deal with the case $N \geq 3$ we introduce an additionnal assumption of stability.

Theorem 5.9 *Assume $N \geq 3$. Let $V \in L^\infty_{loc}(\Omega)$, $V \geq 0$ such that*

$$\lim_{\substack{E \text{ Borel}\\ |E|\to 0}} \int_E V(y)\frac{(\rho(y)-\epsilon)^2_+}{|y-z|^N}dy = 0 \quad \textit{uniformly with respect to } z \in \Sigma_\epsilon \textit{ and } \epsilon \in (0,\epsilon_0]. \tag{5.25}$$

If u is a positive solution of (5.3), then $\mathcal{R}(u) = \partial\Omega$.

Proof. We proceed as in Theorem 5.8. All the relations (5.10)-(5.20) are valid and (5.21) has to be replaced by

$$\int_{\Omega'_{\epsilon_1}\cap E} \chi_{\Omega_{\epsilon_n}}(y)G^{\Omega_{\epsilon_n}}(x,y)V(y)v_{\epsilon_n}(y)dy \leq cc(x)\max_{z\in\Sigma_{\epsilon_n}}\int_{\Omega'_{\epsilon_1}\cap E}\chi_{\Omega_{\epsilon_n}}(y)\frac{\rho^2_{\epsilon_n}(y)V(y)}{|y-z|^{N+1}}dy. \tag{5.26}$$

Since (5.22) is no longer valid, (5.22) is replaced by

$$\int_{\Omega'_{\epsilon_1}\cap E} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y) V(y) v_{\epsilon_n}(y) dy \leq c c(x) \max_{z\in\Sigma_{\epsilon_n}} \int_E V(y) \frac{(\rho(y)-\epsilon_n)_+^2}{|y-z|^{N+1}} dy. \tag{5.27}$$

By (5.25) the left-hand side of (5.27) goes to zero when $|E| \to 0$, uniformly with respect to ϵ_n. This implies that (5.29) is still valid and the conclusion of the proof is as in Theorem 5.8. □

Remark. A simpler statement which implies (5.25) is the following.

$$\lim_{\delta\to 0} \int_0^\delta \left(\int_{B_r(z)} V(y)(\rho(y)-\epsilon)_+^2 dy \right) \frac{dr}{r^{N+1}} = 0, \tag{5.28}$$

uniformly with respect to $0 < \epsilon \leq \epsilon_0$ and to $z \in \Sigma_\epsilon$. The proof is similar to the one of Proposition 2.7.

Remark. When the function V depends essentially of the distance to $\partial\Omega$ in the sense that

$$|V(x)| \leq v(\rho(x)) \qquad \forall x \in \Omega, \tag{5.29}$$

and v satisfies

$$\int_0^a tv(t)dt < \infty, \tag{5.30}$$

Marcus and Véron proved [18, Lemma 7.4] that $\mathcal{R}(u) = \partial\Omega$, for any positive solution u of (5.3). This assumption implies also (5.25). The proof is similar to the one of Proposition 2.8.

5.3 The sweeping method

This method introduced in [21] for analyzing isolated singulariities of solutions of semilinear equations has been adapted in [15] and [19] for defining an extended trace of positive solutions of differential inequalities in particular in the super-critical case. Since the boundary trace of a positive solutions of (5.3) is known on $\mathcal{R}(u)$ we shall study the sweeping with measure concentrated on the singular set $\mathcal{S}(u)$

Proposition 5.10 *Let $u \in C(\Omega)$ be a positive solution of* (5.3) *with singular boundary set $\mathcal{S}(u)$. If $\mu \in \mathfrak{M}_+(\mathcal{S}(u))$ we denote $v_\mu = \inf\{u, u_\mu\}$. Then*

$$-\Delta v_\mu + V(x) v_\mu \geq 0 \qquad \text{in } \Omega, \tag{5.31}$$

and v_μ admits a boundary trace $\gamma_u(\mu) \in \mathfrak{M}_+(\mathcal{S}(u))$. The mapping $\mu \mapsto \gamma_u(\mu)$ is nondecreasing and $\gamma_u(\mu) \leq \mu$.

Proof. We know that (5.31) holds But $Vu_\mu \in L^1_\rho(\Omega) \Longrightarrow Vv_\mu \in L^1_\rho(\Omega)$, if we set $w := \mathbb{G}[Vv_\mu]$, then $v_\mu + w$ is nonegative and super-harmonic, thus it admits a boundary trace in $\mathfrak{M}_+(\partial\Omega)$ that we denote by $\gamma_u(\mu)$. Clearly $\gamma_u(\mu) \leq \mu$ since $v_\mu \leq u_\mu$ and $\gamma_u(\mu)$ is nondeacreasing with μ as $\mu \mapsto u_\mu$ is. Finally, since v_μ is a supersolution, it is larger that the solution of (5.3) with the same boundary trace $\gamma_u(\mu)$, and there holds

$$u_{\gamma_u(\mu)} \leq v_\mu. \tag{5.32}$$

Proposition 5.11 *Let*

$$\nu_{S}(u) := \sup\{\gamma_u(\mu) : \mu \in \mathfrak{M}_+(\mathcal{S}(u))\}. \tag{5.33}$$

Then $\nu_{S}(u)$ is a Borel measure on $\mathcal{S}(u)$.

Proof. We borrow the proof to Marcus-Véron [19], and we naturally extend any positive Radon measure to a positive bounded and regular Borel measure by using the same notation. It is clear that $\nu_{S}(u) := \nu_{S}$ is an outer measure in the sense that

$$\nu_{S}(\emptyset) = 0, \quad \text{and } \nu_{S}(A) \leq \sum_{k=1}^{\infty} \nu(A_k), \quad \text{whenever } A \subset \bigcup_{k=1}^{\infty} A_k. \tag{5.34}$$

Let A and $B \subset \mathcal{S}(u)$ be disjoint Borel subsets. In order to prove that

$$\nu_{S}(A \cup B) = \nu_{S}(A) + \nu_{S}(B), \tag{5.35}$$

we first notice that the relation holds if $\max\{\nu_{S}(A), \nu_{S}(B)\} = \infty$. Therefore we assume that $\nu_{S}(A)$ and $\nu_{S}(B)$ are finite. For $\varepsilon > 0$ there exist two bounded positive measures μ_1 and μ_2 such that

$$\gamma_u(\mu_1)(A) \leq \nu(A) \leq \gamma_u(\mu_1)(A) + \varepsilon/2$$

and

$$\gamma_u(\mu_2)(B) \leq \nu(B) \leq \gamma_u(\mu_2)(B) + \varepsilon/2$$

Hence

$$\begin{aligned} \nu_{S}(A) + \nu_{S}(B) &\leq \gamma_u(\mu_1)(A) + \gamma_u(\mu_2)(B) + \varepsilon \\ &\leq \gamma_u(\mu_1 + \mu_2)(A) + \gamma_u(\mu_1 + \mu_2)(B) + \varepsilon \\ &= \gamma_u(\mu_1 + \mu_2)(A \cup B) + \varepsilon \\ &\leq \nu_{S}(A \cup B) + \varepsilon. \end{aligned}$$

Therefore ν_{S} is a finitely additive measure. If $\{A_k\}$ $(k \in \mathbb{N})$ is a sequence of of disjoint Borel sets and $A = \cup A_k$, then

$$\nu_{S}(A) \geq \nu_{S}\left(\bigcup_{1\leq k\leq n} A_k\right) = \sum_{k=1}^{n} \nu_{S}(A_k) \Longrightarrow \nu_{S}(A) \geq \sum_{k=1}^{\infty} \nu_{S}(A_k).$$

By (5.34), it implies that ν_{S} is a countably additive measure. □

Definition 5.12 *The Borel measure $\nu(u)$ defined by*

$$\nu(u)(A) := \nu_{S}(A \cap \mathcal{S}(u)) + \mu_u(A \cap \mathcal{R}(u)), \qquad \forall A \subset \partial\Omega,\ A \text{ Borel}, \tag{5.36}$$

is called the extended boundary trace of u, denoted by $Tr^e(u)$.

Proposition 5.13 *If $A \subset \mathcal{S}(u)$ is a Borel set, then*

$$\nu_{S}(A) := \sup\{\gamma_u(\mu)(A) : \mu \in \mathfrak{M}_+(A)\}. \tag{5.37}$$

Proof. If $\lambda, \lambda' \in \mathfrak{M}_+(\mathcal{S}(u))$

$$\inf\{u, u_{\lambda+\lambda'}\} = \inf\{u, u_\lambda + u_{\lambda'}\} \leq \inf\{u, u_\lambda\} + \inf\{u, u_{\lambda'}\}.$$

Since the three above functions admit a boundary trace, it follows that

$$\gamma_u(\lambda + \lambda') \leq \gamma_u(\lambda) + \gamma_u(\lambda').$$

If A is a Borel subset of $\mathcal{S}(u)$, then $\mu = \mu_A + \mu_{A^c}$ where $\mu_A = \chi_E \mu$. Thus

$$\gamma_u(\mu) \leq \gamma_u(\mu_A) + \gamma_u(\mu_{A^c}),$$

and

$$\gamma_u(\mu)(A) \leq \gamma_u(\mu_A)(A) + \gamma_u(\mu_{A^c})(A).$$

Since $\gamma_u(\mu_{A^c}) \leq \mu_{A^c}$ and $\mu_{A^c}(A) = 0$, it follows

$$\gamma_u(\mu)(A) \leq \gamma_u(\mu_A)(A).$$

But $\mu_A \leq \mu$, thus $\gamma_u(\mu_A) \leq \gamma_u(\mu)$ and finally

$$\gamma_u(\mu)(A) = \gamma_u(\mu_A)(A). \tag{5.38}$$

If $\mu \in \mathfrak{M}_+(A)$, $\mu = \mu_A$, thus (5.37) follows. □

Proposition 5.14 *There always holds*

$$\nu(u)(Z_V^*) = 0, \tag{5.39}$$

where Z_V^* is the vanishing set of $K_V^\Omega(x, .)$ defined by (4.15).

Proof. This follows from the fact that for any $\mu \in \mathfrak{M}_+(\partial\Omega)$ concentrated on Z_V^*, $u_\mu = 0$. Thus $\gamma_u(\mu) = 0$. If μ is a general measure, we can write $\mu = \chi_{Z_V^*}\mu + \chi_{(Z_V^*)^c}\mu$, thus $u_\mu = u_{\chi_{(Z_V^*)^c}\mu}$. Because of (5.32)

$$\gamma_u(\mu)(Z_V^*) = \gamma_u(\chi_{(Z_V^*)^c}\mu)(Z_V^*) \leq (\chi_{(Z_V^*)^c}\mu)(Z_V^*) = 0,$$

thus (5.39) holds. □

Remark. This process for determining the boundary trace is ineffective if there exist positive solutions u in Ω such that

$$\lim_{d(x)\to 0} u(x) = \infty.$$

This is the case if $\Omega = B_R$ and $V(x) = c(R - |x|)^{-2}$ $(c > 0)$. In this case $K_V^\Omega(x, .) \equiv 0$. For any $a > 0$, there exists a radial solution of

$$-\Delta u + \frac{cu}{(R - |x|)^2} = 0 \qquad \text{in } B_R \tag{5.40}$$

under the form

$$u(r) = u_a(r) = a + c\int_0^r s^{1-N}\int_0^s u(t)\frac{t^{N-1}dt}{(R-t)^2}. \tag{5.41}$$

Such a solution is easily obtained by fixed point, $u(0) = a$ and the above formula shows that u_a blows up when $r \uparrow R$. We do not know if there a exist non-radial positive solutions of (5.40). More generaly, if Ω is a smooth bounded domain, we do not know if there exists a non trivial positive solution of

$$-\Delta u + \frac{c}{d^2(x)}u = 0 \qquad \text{in } \Omega. \tag{5.42}$$

Theorem 5.15 *Assume $V \geq 0$ and satisfies* (2.19). *If u is a positive solution of* (5.3), *then $Tr^e(u) = \nu(u)$ is a bounded measure.*

Proof. Set $\nu = \nu(u)$ and asssume $\nu(\partial\Omega) = \infty$. By dichotomy there exists a decreasing sequence of relatively open domains $D_n \subset \partial\Omega$ such that $\overline{D}_n \subset D_{n-1}$, $\operatorname{diam} D_n = r_n \to 0$ as $n \to \infty$, and $\nu(D_n) = \infty$. For each n, there exists a Radon measure $\mu_n \in \mathfrak{M}_+(D_n)$ such that $\gamma_u(\mu_n)(D_n) = n$, and

$$u \geq v_{\mu_n} = \inf\{u, u_{\mu_n}\} \geq u_{\gamma_u(\mu_n)}.$$

Set $m_n = n^{-1}\gamma_u(\mu_n)$, then $m_n \in \mathfrak{M}_+(D_n)$ has total mass 1 and it converges in the weak*-topology to δ_a, where $\{a\} = \cap_n D_n$. By Theorem 2.6, u_{m_n} converges to u_{δ_a}. Since $u \geq nu_{m_n}$, it follows that

$$u \geq \lim_{n\to\infty} nu_{m_n} = \infty,$$

a contradiction. Thus ν is a bounded Borel measure (and thus outer regular) and it corresponds to a unique Radon measure. □

Remark. If $N = 2$, it follows from Theorem 5.8 that $u = u_\nu$ and thus the extended boundary trace coincides with the usual boundary trace. The same property holds if $N \geq 3$, if (5.25) holds.

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